التحليل الرياضي1

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كلية التربية للعلوم الصرفة
قسم الرياضيات

د. نادية علي ناظم

LCH (1)

Axioms of real numbers

1. The axioms arithmetics
2. The axioms of ordered
3. The complete Axioms

* Let R be a real number and $a, b, c \in R$. Then

A_1 : $\forall a, b, c \in R \;\; a + (b + c) = (a + b) + c$.

A_2 : $a + b = b + a$

A_3 : $for\ any\ a \in R, \exists!\ element\ 0 \in R\ \ s.t$

$a + (-a) = -a + a = 0$

A_4 : $Ther\ exists\ an\ element\ 0 \in R\ , S.t$

$a + 0 = 0 + a = a$

Then $(\boldsymbol{R}, +)$ is a commutative group.

A_5 : $a.(b.c) = (a.b).c$

A_6 : $a.b = b.a$

A_7 : $\exists!$ Element in $R(1 \in R\ )\ s.t\ a.1 = 1.a = a$

A_8 : $\forall a \in R\ , \exists!\ a^{-1} \in R\ , s.t\ a.a^{-1} = a^{-1}.a = 1$

Form $\boldsymbol{A_5} \rightarrow \boldsymbol{A_8}$. $(\boldsymbol{R}, .)$ commutive ring

A_9 : $a.(b + c) = (a.b) + (a.c)$

$A_1 \rightarrow A_9$ $(R, +, .)$ Is a field

Def:

$*$ Subtraction $a - b = a + (-b)\ , \forall a, b \in R$

$*$ Division $a \div b = a.b^{-1} \ni b \neq 0$

The Axioms of order:

$A_{10}: a \leq b \ or \ b \leq a$

$A_{11}: a \leq b \ and \ b \leq c \rightarrow a = b$

$A_{12}: a \leq b \ and \ b \leq c \rightarrow a \leq c$

$A_{13}: a \leq b\ , c \in R \rightarrow a + c \leq b + c$

$A_{14}: a \leq b\ , c \ is\ not\ negative \rightarrow a.c < -b.c$

$A_1 \rightarrow A_{14}$, $(R, +\ , .\ , \leq)$ order field.

Remark:

$R^{+} = \{x \in R\ ; x > 0\}$

$R^{-} = \{x \in R\ ; x < 0\}$

Propositions: Let $(R, +, .)$ be a field, then prove the following

1. $\forall a, b, c \in R\ , if\ a + b = b + c\ , then\ a = c$
2. $\forall a, b, c \in R\ , if\ a.b = c.b\ \ , then\ a = c$
3. $\forall a, b \in R$,prove that:

1. $-(-a) = a$
2. $(a^{-1})^{-1} = a$
3. $(-a) + (-b) = -(a + b)$
4. $(-a).b = -a, b$
5. $if\ a.b = 0$ then either $a = 0\ or\ b = 0$

Proof (5):

Let $a \neq 0$, T.P $b = 0$

Since $a \neq 0$, then $\exists a^{-1} \in R\ s.t\ \ a.a^{-1} = 1$

$a^{-1}(a.b) = 0$

$(a^{-1}.a).b = 0$

$1.b = 0 \to b = 0$

Let $b \neq 0$, T.P $a = 0$

Since $b \neq 0$, then $\exists b^{-1} \in R\ s.t\ \ b.b^{-1} = 1$

$(a.b)b^{-1} = 0$

$a.(b.b^{-1}) = 0$

$a.1 = 0 \to a = 0$

Absolute Value:

let $a \in R$, the absolute value of a is:

$$|a| = \begin{cases} a & if\ a > 0 \\ 0 & if\ a = 0 \\ -a & if\ a < 0 \end{cases}$$

$|a|: R \to R^{+} \cup \{0\}$ is the function of absolute value.

Properties of absolute value.

Theorem: let a be a real number, then

1. $|x| < a \ \leftrightarrow -a < x < a$

2. $|X| > a \leftrightarrow x > a \text{ or } x < -a$

Corollary: let $a \in R^+$ and $b \in R$, then

1. $|x - b \leq a \; iff \; b - a \leq x \leq b + a$
2. $|x - b| \geq a \; iff \; x \geq b + a \;\; or \;\; x \leq b - a$

Let $a, b \in R$ and k be areal number, then

1. $|a| \geq 0$
2. $|a| = 0$ iff $a = 0$
3. $a^2 = |a|^2$
4. $|ab| = |a|.|b|$
5. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
6. $|ka| = |k|.|a|$

Example: $\forall a \in R \, , \sqrt{a^2} = |a|$

Proof:

If $a > 0$ then $\sqrt{a^2} = a$

If $a < 0$ then $\sqrt{a^2} = -a$

by def absolute value to a we have

$$|a| = \begin{cases} a = \sqrt{a^2} \ \ if \ \ a \geq 0 \\ -a = \sqrt{a^2} \ \ if \ \ a < 0 \end{cases}$$

وفي كلتا الحالتين يكون لدينا $|a| = \sqrt{a^2}$

The triangle inequality

Theorem: if $a, b \in R$, then $|a + b| \leq |a| + |b$

Proof:

$$|a + b|^2 = (a + b)^2 \leq a^2 + 2ab + b^2$$
$$\leq |a|^2 + 2|ab| + |b|^2$$
$$\leq (|a| + |b|)^2$$
$$\therefore |a + b| \leq |a| + |b|$$

Corollary: if $a, b \in R$, then $|a - b| \geq |a| - |b|$

LCH (2)

Def: let $S \subset R$ S is said to be bounded above if there is some real numbers m s.t $x \leq m \ \forall x \in S$, m is called upper bounded of S

LCH (3)

Proposition:

If $\emptyset \neq S \subset R$ and $\sup(S) = M$, then $\forall p < M \;\; \exists x \in S$ s.t $p < x \leq M$

i.e.: if $\sup(S) = M$ then $\forall \epsilon > 0 \, , \exists x \in S$ s.t $M - \epsilon < x \leq M$

proof:

let $\sup(S) = M$ then $\forall x \in S, \;\; x \leq M$

T.P $\;\; \forall x \in S \, , p < x$?

Suppose that $x \leq p \, , \forall x \in S$

$\rightarrow$ p is upper bounded for S, but by hypothesis

$p < M = \sup(S)$ .... C!

$\therefore \;\; \exists \, x \in S \ni p < x \leq M$.

Theorem: The set N of natural numbers is unbounded above in R

Proof:

Suppose N is bounded above.

By completeness axiom

N has a supreme M

Let $\sup(N) = M$

From proposition above $\exists n \in N$ s.t $M - 1 < n < M$.

Then $M - 1 < n \rightarrow M < n + 1$,
But $n + 1 \in N$
And $n + 1 > M = \sup(N) \rightarrow C!$
Therefore, N is unbounded above

Theorem: Archimedan property

If $x \in R^{++}$ then for any $y \in R$, there exists $n \in N$ s.t $n > y$

Def: let F a field, F is called Archimedean filed, if for any $x \in F, \exists n \in N$ s.t $n > x$

i.e.: N is abounded above in F

Ex:

1. R is Archimedean field
2. Q is Archimedean field
3. $s = \{a + b\sqrt{2} : a, b \in Q\}$ is Archimedean field

Theorem: Denseness property

Between any two distinct reals, there exists infinitely many rationales and irrationals

LCH (4)

Def: (irrational numbers Q')

Let Q' be a complement of Q in the real number R.

i.e.: $Q' = R - Q$, we called is set of irrational numbers

remark: $R = Q \cup Q'$

Theorem: prove that $\sqrt{2}$ is irrational number

i.e.: There are no rational numbers whose square is 2

i.e.: $\nexists\, x \in Q \ni x^2 = 2$

proof:

suppose $\sqrt{2}$ is rational number i.e. $\sqrt{2} = \frac{m}{n}$

So $2 = \frac{m^2}{n^2}$, then $m^2 = 2n^2$

Case 1:

m and n are odd.

Since m is odd → m^2 is odd

Since n is odd → n^2 is odd

But $2n^2$ is even → $m^2 = 2n^2 \rightarrow C!$

Case 2:

m is even and n is odd, then $m = 2p$

and $m^2 = 4p^2$, → $4p^2 = 2n^2 \rightarrow 2p^2 = n^2 \rightarrow C!$

Case 3:

m is odd and n is even, then, since m is odd → m^2 is odd, and $2n^2$ is even → $m^2 = 2n^2 \rightarrow C!$

∴ $\sqrt{2}$ is irrational number

Theorem: Q is not Complete field

Theorem: for every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$

i.e.:$\forall x > 0 , \forall n \in N , \exists! , y \in R^+ \ s.t \ y = \sqrt[n]{x}$

Theorem: if $\frac{m}{n}$ and $\frac{p}{q}$ are rationales and $q \neq 0$ then $\frac{m}{n} + \sqrt{2}\frac{p}{q}$ is irrational number

Proof:

Suppose $\frac{m}{n} + \sqrt{2}\ \frac{p}{q}$ is rational

Then there is $r, s \in Z , s \neq 0 \ s.t \ \frac{m}{n} + \sqrt{2}\frac{p}{q} = \frac{r}{s}$

So $\sqrt{2}\frac{p}{q} = \frac{r}{s} - \frac{m}{n} \rightarrow \sqrt{2} = \frac{p}{q}\left(\frac{rn-sm}{sn}\right) \in Q$

So $2 = \left(\frac{q(nr-sm)}{psn}\right)^2 \rightarrow !$ with theorem: $\nexists\, x \in Q \ni x^2 = 2$

Theorem: Between any two distinct rationales there is an irrational number.

LCH (5)

Ex:

1.Prove $x^2 \geq 0 , \forall\, x \in R$

2.Let a, b be tow real s.t $a \leq b + \epsilon\ \forall\ \epsilon > 0$ then $a \leq b$

Proof (2):

Suppose $a > b$

Then $a + a > b + a$

$\frac{2a}{2} > \frac{b+a}{2}$

$a > \frac{b+a}{2}$ ………….(1)

Take $\epsilon = \frac{a-b}{2} > 0$ (Since $a > b$, then $a - b > 0 \rightarrow \frac{a-b}{2} > 0$)

$a \leq b + \epsilon \rightarrow a \leq\ b + \frac{a-b}{2} = \frac{2b+a-b}{2} = \frac{a+b}{2} < a$

From (1) ……………. C!

$a \leq b$

Ex:

1.Q is order field ($A_1 \rightarrow\ A_{14}$)

2.C is field but not order

since: if $x = 1 \rightarrow x = \sqrt{1} \rightarrow x^2 = -1 < 0 \rightarrow C!$

since: ($x^2 \geq 0 , \forall x \in R$)

Metric space

Def: let X be anon-empty set and $d: X \times X \to R^+$ be a mapping. We say that order (X, d) is metric space if it is satisfying the following:

1. $d(x, d) \geq 0, \forall x, y \in X$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$
4. $d(x, y) = 0 \leftrightarrow x = y$

Not: d is called metric mapping
$d(x, y)$ is a distance between x and y

Remark: A mapping $d: X \times X \to R^+$ is called a pseudo metric for X iff d satisfies (1,2,3) in the above definition and $d(x, x) = 0, \forall x \in X$

Cauchy - Shwarz inequality
Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots b_n)$ be two tripe of complex number , then:

$$\sum_{i=1}^{n} |a_i + b_i| \leq \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}} . \left(\sum_{i=1}^{n} |b_i|^2\right)^{\frac{1}{2}}$$

Minkowskis inequality

$$\left(\sum_{i=1}^{n}|a_i+b_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|b_i|^p\right)^{\frac{1}{p}} \quad ,p \ge 1$$

Ex: if $X = R\ and\ d(x,y) = |x-y|$, show that (X,d) is a metric space.

Solution:

1. $d(x,y) = |x-y| \ge 0$ by def. of Absolute value
2. $d(x,y) = |x-y| = |-(y-x)| = |y-x| = d(y,x)$
3. $d(x,z) = |x-z| = |x-y+y-z|$
   $\le |x-y| + |y-z|$
   $= d(x,y) + d(y,z)$
4. $d(x,y) = 0$ iff $x = y$
   $d(x,y) = 0$ iff $|x-y| = 0$
   iff $x - y = 0$
   iff $x = y$

$\therefore (X,d)$ is a metric space

Discrete metric space

Let $X \neq \emptyset$ and $d: X \times X \to R$ s.t

$$d(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}$$

$\forall x,y$, show that (X,d) is metric space

Solution:

1.$d(x,y) \geq 0$, $\forall x,y \in X$ (by def. d)

2.$d(x,y) = d(y,x)$?

if $x = y \rightarrow d(x,y) = 0 = d(x,y)$

if $x \neq y \rightarrow d(x,y) = 1 = d(y,x)$

3.Let $x,y,z \in X$ T.P $d(x,y) \leq d(x,y) + d(y,z)$?

if $x = z$ then $d(x,z) = 0$

since $d(x,y) \geq 0$ and $d(y,z) \geq 0$ then

$$d(x,z) \leq d(x,y) + d(y,z)$$

if $x \neq z$ then $d(x,z) = 0$

since $d(x,z) = 1$ and either $x \neq y$ or $x \neq z$, $y = z$

either: $d(x,z) = d(x,y) = d(y,z) = 1$

or: $d(x,z) = d(x,y) = 1$ and $d(y,z) = 0$

then: $d(x,z) \leq d(x,y) + d(y,z)$

$$1 \leq 1 + 1$$

$$1 \leq 1 + 0$$

LCH (6)

Ex: show that (X,d) is pseudo metric space but not metric where

$d: X \times X \to R$, $d(x, y) = |x^2 - y^2|$, $forall\ x, y \in R$.

Solution:

Let $x, y, z, \in R$

1- $d(x, y) = |x^2 - y^2| \geq 0$, by def Abs. Value

2- $d(x, y) = |x^2 - y^2| = |-(y^2 - x^2)| = |y^2 - x^2| = d(y, x)$

3- $d(x, y) = |x^2 - y^2| = |x^2 - z^2 + z^2 - y^2| \leq |x^2 - z^2| + |z^2 - y^2|$

$\leq d(x, z) + d(z, y)$

4- $d(x, x) = |x^2 - x^2| = 0$, $\forall x \in R$

$\therefore (X, d)$ pseudo metric space but not metric space,

since, if $d(x, y) = 0 \to |x^2 - y^2| = 0 \to x^2 - y^2 = 0 \to x^2 = y^2$

$\to x = y$

ex: let $x = 1$, $y = -1$

then $d(x, y) = d(1, -1) = |1^2 - (-1)^2| = 0$, but $1 \neq -1$

Def: let (X, d) be a metric space $S, T \subseteq X$, $p \in S$ then

1- The distance between p and S is
$d(p,S) = \inf\{d(p,x) : x \in S\}$
2- The distance between S and T is
$d(S,T) = \inf\{d(x,y) : x \in S , y \in T \}$
3- Diameter of S is $d(S) = \sup\{d(x,y) : x,y \in S\}$
4- S is called bounded, if $\exists\ M \in R^{++}$, s.t $d(x,y) \leq M\ , \forall x,y \in S$.

Def: let (X,d) be a metric space and $S \subseteq X$, S is called open set, if $\forall x \in S\ , \exists r > 0$ s.t $B(x,r) \subset S$

Ex: if $S = \emptyset$, then S is open set
If $x \in S \rightarrow \exists r > 0\ \ s.t\ \ B(x,r) \subset S$
$F \rightarrow F \quad or \quad T \ : \ T$

LCH (7)

If S = X , then S is open set
Solution:
Since all balls is contains in X

Any open interval is open set. But the convers is not true
Solution:

Let $x \in s \rightarrow x \in (a,b) \subseteq (a,b) = S.$

So. S is open set

Ex: Let $S = (-1,1) \cup (2,3)$

Let $x \in s$, then $x \in (-1,1)\ or\ x \in (2,3)$

Then $x \in (-1,1) \subset S\ or\ x \in (2,3) \subset S$

∴ S is open set. But is not open interval

Any ball is open set.

Proof:

$\forall y \in B(x,r), \exists w > 0, s.t\ B(y,w) \subset B(x,r)$?

Let $w = r - d(x,y) > 0$

Let $Z \in B(y,w) \longrightarrow d(z,y) < w$

$d(Z,y) \leq d(x,y) + d(y,z)$

$\leq d(x,y + w$

$= d(x,y) + r - d(x,y)$

$= r$

Then $Z \in B(x,r) \longrightarrow B(y,w) \subset B(x,r)$

This is true for all y in B(x,r)

So B(x,r) is open set

$\boldsymbol{S = \{x\}, x \in R}$ **is not open set**

Since there is not open interval in S Containing x and Contained in S i.e (($\forall r > 0, \exists B(x,r) = (x-r, x+r) \subset S$))

$[\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{a}, \boldsymbol{b}), [\boldsymbol{a}, \infty)$ *and* $(-\infty, \boldsymbol{b}]$ are not open set

Proof:

If S=[a,b] ,then S is not open set ?

Since, if $x = a \rightarrow \forall r > 0 , B(a,r) = (a-r, a+r) \not\subset [a,b]$

The intersection of any tow open set is open set

i.e ((the intersection of any finite family of open set is open))

Proof:

Let $A = \{ S_k : S_k \text{ is open set } k = 1,2,\ldots,n\}$

$T.p \ \bigcap_{k=1}^{n} S_k$ is open set

Let $x \in \bigcap_{k=1}^{n} S_k \rightarrow x \in S_k, \forall k$, but S_k is open set $\forall k$, then $\exists r_k > 0$ s.t $B(x, r_k) \subset S_k$

Let $r = \min\{r_1, r_2, \ldots, r_n\}$

Then $B(x,r) \subset S_k, \forall k$.

$\therefore B(x,r) \subset \bigcap_{k=1}^{n} S_k$, therefore $\bigcap_{k=1}^{\infty} S_k$ is open set.

Theorem: the infinite intersection of open sets is not necessary open set.

Ex: let $S_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ $\forall x \in R$, open interval.

$n = 1 \rightarrow s_1 = (x - 1, x + 1)$

$n = 2 \rightarrow S_2 = (x - \frac{1}{2}, x + \frac{1}{2})$

$n = 3 \rightarrow S_3 = (x - \frac{1}{3}, x + \frac{1}{3})$

. .

. .

When $n \rightarrow \infty \bigcap_{k=1}^{\infty} S_k = \{x\}$ is not open

Theorem: the union of any family (finite or infinite) – (countable or uncountable) of open set is open

Proof:

Let $A = \{S_\alpha , S_\alpha \text{ is open set } \alpha \in \wedge\}$

T.P: $\bigcup_{\alpha \in \wedge} S_\alpha$ is open set

Let $x \in \bigcup_{\alpha \in \wedge} S_\alpha \rightarrow \exists \alpha \in \wedge \; s.t \; x \in S_\alpha$

Since S_α is open set $\rightarrow \exists \alpha > 0 \; s.t$

$B(x, r_\alpha) \subset S_\alpha$, then $x \in B(x, r_\alpha) \subset S_\alpha \subset \bigcup_{\alpha \in \wedge} S_\alpha$

This is true $\forall x \in \bigcup_{\alpha \in \wedge} S_\alpha$, therefore $\bigcup_{\alpha \in \wedge} S_\alpha$ is open set

Theorem: S is open iff S is the Union of balls

LCH (8)

Def: let X be anon-empty set and τ is a family of subsets of X, if τ satisfy the following

1- $\phi, X \in \tau$
2- If $G, H \in \tau \rightarrow G \cap H \in \tau$
3- If $\{G_\lambda\} \in \tau \rightarrow \bigcup_{\lambda\in\Lambda} G_\lambda \in \tau$

Then, the order pair (X,τ) is called topological Space.

Theorem: every metric space is topological space.

Proof:

Let (X,d) be a metric space and $\tau =$ the family of all open subsets of X, then

1- ϕ, X open sets $\rightarrow \phi, X \in \tau$

2- $G_1, G_2 \in \tau \rightarrow G_1, G_2$ are open sets

$\rightarrow G_1 \cap G_2 \in \tau$

3- If $G_\lambda \in \tau, \lambda \in\Lambda \rightarrow \forall\lambda, G_\lambda$ open subset of X

$\rightarrow \bigcup_{\lambda\in\Lambda} G_\lambda$ open set of

$\rightarrow \bigcup_{\lambda\in\Lambda} G_\lambda \in \tau$

$\therefore (X,\tau)$ is a topological space

Def: let $d_1$and d_2 be two metric mapping in the set X, then d_1, d_2 are called Equivalent if every open set in (X,d_1) is open in (X,d_2) and Vice Versa

Def: let (X, d) be a metric space and $S \subseteq X$, S is called closed set if S^c is open Set where $S^c = X - s$ (Complement of S)

Ex:

1- $S = X$ is closed set.

   Solution:

   Since $S^c = X^c = \phi$ open set

2- $S = \phi$ is closed set

   Solution:

   since $S^c = \phi^c = X$ is open set

3- $S = [a, b], [a, b), S = (-\infty, b]$ are closed set in R

   Solution:

   if $S = [a, b] \to S^c = (-\infty, a) \cup (b, \infty)$ open set $\to S$ is closed set

4- In R , $S = \{x\}$ is closed set

   Since :

   $S^c = (-\infty, x) \cup (x, \infty) \to S^c$ is open, So S is closed set.

5- Any finite set in R is closed set

   Solution:

   let $S = \{x_1, x_2, \ldots, x_n\} \subseteq R$.

   $S^c = (-\infty, x_1) \cup (x_1\ , x_2) \cup \ldots \cup (x_{n-1}\ , x_n)\ \cup$

(x_n, ∞)

So, S^c is open , then S is closed set

6- If $S = N$, $S = Z$, then S is Closed set

Solution:

let $S = N$

then $S^c = (-\infty, 1) \cup (1,2) \cup (2,3) \ldots (\bigcup_{n=4}^{\infty}(n, n+1))$

$\rightarrow S^c$ is open $\rightarrow$ S is closed

if $S = Z \rightarrow S^c = (\bigcup_{n=1}^{\infty}(-(n+1), -n)) \cup (-1,0) \cup (0,1) \cup (\bigcup_{n=1}^{\infty}(n, n+1))$

S^c is open, then S is closed

LCH (9)

7- The Union of finite number of closed sets is closed.

Solution:

let $A = \{S_i, ; \; S_i \; closed \; set \; in \; X \, , i = 1,2, \ldots, n\}$

T.P: $\bigcup_{i=1}^{n} S_i$ is closed set

i.e. T.P $(\bigcup_{i=1}^{n} S_i)^c$ is open set

Since S_i is closed, $\forall i$ then S_i^c is open $\forall i$

and $\bigcap_{i=1}^{n} S_i^c$ is open

So, $(\bigcup_{i=1}^{n} S_i)^c$ is open $\quad ((\bigcup_{i=1}^{n} S_i)^c = \bigcap_{i=1}^{n} S_i^c)$

therefore $\bigcup_{i=1}^{n} S_i$ is closed.

Remark: the infinite union of closed sets is not necessary closed set

Ex: let $S_n = \left\{ \left[\frac{-n}{n+1}, \frac{n}{n+1}\right] : n \in N \right\}$, S_n is closed interval, Is $\bigcup_{n=1}^{\infty} S_n$ is closed?

Solution:

If $n = 1 \rightarrow S_1 = \left[\frac{-1}{2}, , \frac{1}{2}\right]$

If $n = 2 \rightarrow S_2 = \left[\frac{-2}{3}, \frac{2}{3}\right]$

. .

. .

When $n \rightarrow \infty \Longrightarrow \lim_{n\rightarrow\infty} \frac{\pm n}{n+1} = \lim_{n\rightarrow\infty} \frac{\pm\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \pm 1$

$\therefore \bigcup_{n=1}^{\infty} s_n = (-1,1)$ open set

Theorem: The infinite intersection of closed set S is closed?

Def: let X be a metric space and $S \subseteq X, p \in X$, p is called an accumulation point of S if every open set contain p , contains another point q s.t $p \neq q, q \in S$.

i.e.: p is a cc. point of S if $\forall U$, U is open set $p \in U$, then $U - P \cap S \neq \phi$

Remark: Since every open set is Union balls. So, we can define acc. Point as following:

P is acc. Point of S, if $\forall r > 0\ B(p,r) - \{p\} \cap S \neq \phi$

* S' is the closure of all acc. Point of S (Derived set)
* $\overline{S}$ is the closure of S and $\overline{S} = S \cup S'$
* P is not acc. Point, if $\exists U$, U is open and $p \in U$ S.t $U - \{p\} \cap S = \phi$. (i.e. $\exists r > 0$, $B(r,p) - \{\ p\ \} \cap S = \phi$

Ex: let $s = \{1,5\}$, find $S'\ and\ \overline{S}$

Solution: TO find S' there are some cases

LCH (10)

$x = 1\ \ ,x = 5\ \ ,x < 1\ \ ,x > 5\ \ ,\ \ 1 < x < 5$

If $x = 1 \rightarrow$ x is not acc. Point since , $\exists r > 0$

$B(x,r) - \{x\} \cap S = \emptyset$, when $r = 1$

$B(1,1) - \{1\} \cap \{1,5\} = (0,2) - \{1\} \cap [1,5\} = \emptyset$

If $x = 5 \rightarrow$ x is not acc. Point, since $\exists r > 0$, $B(x,r) - \{x\} \cap S = \emptyset$, when $r = 1$

$\rightarrow B(5,1) - \{5\} \cap \{1,5\} = (4,6) - \{5\} \cap \{1,5\} = \emptyset$

If $x < 1 \rightarrow$ x are not acc. Point since $x \in (x - 1,1)$ and $(x - 1,1) \cap S = \emptyset$

If $x > 5 \rightarrow x$ are not acc. Point, since $x \in (5, x + 1) and\ (5, x + 1) \cap S = \emptyset$

If $1 < x < 5$ are not acc. Point since,

$x \in (1,5)\ \ and\ \ (1,5) \cap S = \emptyset$

So, S has no a acc. Point then $S' = \emptyset$ and $\overline{S} = S \cup S' = S \cup \emptyset = S$.

Let $s = \left\{1 , \frac{1}{2} , \frac{1}{3} , \ldots \right\} = \left\{\frac{1}{n} , n = 1,2,3, \ldots . \right\}$ show that $S' = \{0\}$

If $S = (a, b)$, find S'

Solution:

If $x = a \rightarrow x$ is acc. Point since $\forall r > 0$,

$a \in B(0, r) = (a - r, a + r)$ and $B(a, r) - \{a\} \cap S \neq \emptyset$

If $x = b \rightarrow x$ is acc. Point, since $\forall r > 0$, $b \in B(b, r)$

$B(b, r) = (b - r, b + r)\ and\ B(b, r) - \{b\} \cap (a, b) \neq \emptyset$

If $a < x < b \rightarrow x$ are acc. Point since $\forall\ r > 0$,

$x \in B(x,r) = (x-r, x+r)$ and $B(x,r) - \{x\} \cap S \neq \emptyset$

That is $(x-r, x+r) - \{x\} \cap (a,b) \neq \emptyset$

If $x < a \rightarrow x$ are not acc. Point since $x \in (x-1, a)$ and $(x-1,a) \cap S = \emptyset$

If $x > b \rightarrow x$ are not acc. Point, since $x \in (b, x+1)$ and $(b, x+1) \cap (a,b) = \emptyset$

$\therefore S' = [a,b] \rightarrow \overline{S} = S \cup S' = [a,b]$

LCH (11)

Def: A sub set A of a metric space X is said to be dense if $\overline{A} = X$

Ex: prove that $\bar{Q} = R$ (i.e., Q dense set in R)

Solution:

If $x \in R$, then x is acc. Point in Q.

Since any open interval Contain x Contains infinitely rational and irrationals

Then $Q' = R$

So $\overline{Q} = Q \cup Q' = Q \cup R = R$

Def: a metric space is called separable if it has a countable dense subset.

Ex: R separable since Q countable and $Q \subseteq R$, with Q dense in R

Theorem: let X be a metric space, $S \subseteq X$ then

1- S is closed iff $S' \subset X$
2- $\overline{S}$ is closed set
3- $\overline{S} = S$ iff S closed set
4- $\overline{S}$ is smallest closed set contains S.

Compact Space

Def: let (X, d) be a metric space, $\emptyset \neq S \subseteq X$, if the set $\{U_\lambda : U_\lambda \ open\ set, \lambda \in \Lambda\}$ is a family of open subsets of X such that $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$, then the family $\{U_\lambda\}$ is called open cover for S in X.

- If the family $\{U_\lambda\}$ is finite and $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$ then $\{U_\lambda\}$ is called finite cover.
- Let $\{U_\lambda\}$ and $\{U_\alpha\}$ be to open cover for S and $U_\lambda \in \{U_\alpha\}\ \forall \lambda$, then $\{U_\lambda\}$ is called subcover for $\{U_\alpha\}$

Def: let A be a subset of a metric space (X, d), A is called compact set if every open cover for A in X has a finite subcover.

Exp: Any finite subset B of matric space (X, d) is **compact set**

Ex: R is not compact

Ex : Any open interval A=(a,b) is not compact

Ex : Any closed interval A=[a,b] is Compact.

Proof :

Since we can restrict any open cover for A to finite subcover such as :

Let $\epsilon > 0, B = \{(a - \epsilon, a + \epsilon, (a, b), (b - \epsilon, b + \epsilon)\}$

(a) (b]

Theorem: ((Bolzano weir strass theorem))
In compact space X, every infinite subset S of X has at least one accumulation point.

Theorem : In compact metric space, every closed subset is compact.

Proof : X be a compact metric space, and A be a closed subset of X, then
A^c is open. T.P A is compact.
Let $B = \{U_\lambda : U_\lambda \text{ is open set in } X, \forall \lambda \in \wedge \}$ be open cover for A.
Then $A \subseteq \bigcup_{\lambda \in \wedge} U_\lambda$
Sine $X = A \cup A^c \subseteq (\bigcup_{\lambda \in \wedge} U_\lambda) \cup A^c$,
But A^c is open set then $\bigcup_{\lambda \in \wedge} U_\lambda \cup A^c$ is open cover for X, since X is compact set , then there exists a finite member $\lambda_1, \lambda_2, \dots \dots, \lambda_n$ such that

$$X = A^c \cup \left(\bigcup_{i=1}^{n} U_{\lambda i} \right)$$

Since that $X = A^c \cup (\bigcup_{i=1}^{n} U_{\lambda i})$ .Since $A \cap A^c = \emptyset$, then $A \subseteq \bigcup_{i=1}^{n} U_{\lambda i}$
$\Rightarrow$ B has a finite subcover $\{ U_{\lambda 1}, U_{\lambda 2}, \dots \dots, U_{\lambda n}\}$.
For A, $\Rightarrow$ A is compact.

LCH (13)

Theorem: Let (X, d) be a metric space, $A \subseteq X$, If A is compact, Then A is closed

Theorem: Let (X, d) be a metric space, $A \subseteq X$, If A is compact, Then A is bounded

Remark: In metric space

Compact $\rightarrow$ Closed + bounded
$\not\leftarrow$

Theorem: Let $\{I_n : n = 1,2,3, \dots\}$ be a family of closed interval
if $I_{n+1} \subset I_n, \forall n, then \ \bigcap_{n=1}^{\infty} I_n = \emptyset$

Theorem: (**Hien-Bord Theorem**)
Every closed and bounded subset of R^n, $n \geq 1$, is compact.

Chapter Three

Sequences in Metric Space

Definition: Let S be any set a function f whose domain is the set N and the range is S is
Called a sequence in S.
i.e. $f: N \rightarrow S$, where $\forall\, n \in N, \exists\, x_n \in S\ s.t\ f(n) = x_n$

1. $< \frac{1}{5n} > = \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \dots$

2.$< \frac{1}{n+1} > = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

3.$< 4 > = 4, 4, 4, \ldots$

4.$< n - 3 > = -2, -1, 0, 1, \ldots$

Def: Let (X, d) be a metric space and $< X_n >$ be seq. in X, then $< X_n >$ is said to be converges to appoint in X, if $\forall \epsilon > 0, \exists k \in N \; s.t \; d?(X_n, x) < \epsilon, \forall n > k$. We write $X_n \to x$ or $\lim_{n\to\infty} X_n = x$, x is called

LCH (14)

A Limit point of $< X_n >$.

If $\forall n > K$, does not Converge, them $< X_n >$ is called divergent Sequence.

Not that: K depend on ϵ only.

التغير الهندسي للتعريف التقارب

$$(X_n \to x)$$

يعني الكرة التي مركزها x ونصف قطرها ϵ تمتلك عدد غير منتهي من حدود او نقاط المتتابعة X_n لانه

$\forall \epsilon > 0, \exists k \in N \; s.t \; d(X_n, x) < \epsilon, \forall n > k \implies X_n \in B(x, \epsilon)$.

Ex: Let $< X_n > = < 1 >$ constant seq. show that $\lim_{n\to\infty} X_n = 1$

$< 1 >$ convergs to 1 since $\forall \epsilon > 0, \exists k \in N$

$s.t \; d(X_n, x) = |1 - 1| = 0 < \epsilon, \forall n > k$

Ex: Let $< X_n >$ be a seq. defined by $X_n = \begin{cases} n \; if \; n \leq 50 \\ \\ 3 \; if \; n \geq 50 \end{cases}$

.show that $\lim_{n\to\infty} X_n = 3$

Solution:

$< X_n \geq 1,2,3, \dots, 50 , 3 , 3 , 3 , \dots$

$\forall \epsilon > 0 , \exists k = 50 \; s.t \; d(X, x) = |3 - 3| = 0 < \epsilon$

Ex: Show that $\lim_{n\to} X_n = 2$, where $< X_n > = < \frac{2n-3}{n+1} >$

Solution:

$\forall \epsilon > 0$, to find $K \in N$ s.t $d(X_n, x) < \epsilon, \forall n > k$?

$$d(X_n, x) = \left|\frac{2n-3}{n+1} - 2\right| = \left|\frac{2n-3-2(n+1)}{n+1}\right|$$

$$= \left|\frac{2n-3-2n-2}{n+1}\right| = \left|\frac{-5}{n+1}\right| = \frac{5}{n+1}$$

$\forall \epsilon > 0,$ by Arch. Property $\rightarrow \exists K \in N \ni$

$\forall k > 5 \rightarrow \frac{5}{\epsilon} < k.$

$\forall n > K \rightarrow n + 1 > k + 1 \; and \; k + 1 > k , k > \frac{5}{\epsilon}$

$\Longrightarrow n + 1 > k + 1 > k > \frac{5}{\epsilon}$

$\frac{1}{n+1} < \frac{\epsilon}{5} , \forall n > k$

Exc:

1.Let $< X_n > = < \frac{2}{\sqrt{n}} >$, show that $\lim_{n\to\infty} X_n = 0$

2.Let $< X_n > = < \frac{5n-4}{2-3n} >$, show that $\lim_{n\to\infty} X_n = -\frac{5}{3}$

3.Let $< X_n > = < \frac{2-7n}{1-5n} >$, show that $\lim_{n\to\infty} X_n = \frac{7}{5}$

Show that the following sequence are divergent

1.$< X_n > = < \sqrt{n} >$

2.$< X_n > = < (-1)^n >$

3.$< X_n > 3^n >$

4.$< X_n > = < \frac{n^2}{2n-1} >$

Theorem: If $< X_n >$ is convergent sequence in (X, d), then $< X_n >$ has a unique limit point.

Proof:

Suppose $< X_n >$ has two limit points x and y with $x \neq y$ and $d(x, y) = \epsilon$

Since $X_n \to y \Longrightarrow \forall \epsilon > 0, \exists k_2 \in N\ s,t\ d(x, y) < \frac{\epsilon}{2}$

Let $k = \max\{k_1, k_2\}$

Since $d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\Rightarrow d(x, y) < \epsilon, \forall \epsilon > 0$

This true only when $d(x, y) = 0 \Rightarrow x = y \to C!$

$\therefore < X_n >$ has a unique limit point.

LCH (15)

Definition: A seq. $< X_n >$ is called bounded the set $\{X_n : n \in N\}$ is bounded

i.e. $< x_n >$ is bounded if

$\exists m > 0 \; s.t \; d(x_n, x_m) \leq M \;, \forall n \;, \forall m.$

Ex:

1.$< \frac{(-1)^{n+1}}{n} > = 1 \,, -\frac{1}{2} \,, \frac{1}{3} \,, -\frac{1}{4} \,, \ldots$

$|x_n| = \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} \leq 1 \;\Longrightarrow < x_n >$ is bounded

and $M = 1$

2.$< 5 + \frac{(-1)^{n+1}}{n} > = 6 \,, \frac{9}{2} , \frac{16}{3} \,, \ldots$

$< x_n \geq 5 + \frac{1}{n} \leq 5 + 1 = 6 \;\Longrightarrow < x_n >$ is bounded

and $M = 6$

$$3.< n + (-1)^n > = \begin{cases} < n - 1 > \;, \; if\; n\; is\; odd \\ < n + 1 > \;, \; if\; n\; is\; even \end{cases}$$

$$4. |x_n| = \begin{cases} |n-1| \geq 0 \\ \\ |n+1| \geq 2 \end{cases}$$

Theorem: In metric space. Every convergent sequence is bounded.

Proof:

Let $< x_n >$ be a convergent sequence in (X, d) and $x_n \to x$, to prove $< x_n >$ is bounded

Since $x_n \to x \implies \forall \epsilon > 0 , \exists k \in N \; s.t \; d(x_n, x) < \epsilon, \forall n > k$

That $\epsilon = 1 \implies d(x_n, x) < 1, \forall n \in k.$

Let $r = \max\{1 , d(x_1, x) , \; d(x_2, x) , \dots , d(x_n, x) \}$

$\implies d(x_n, x) < r$

$\therefore < x_n >$ is bounded and $M = 2r$

Remark: The convers of above theorem is not true.

Ex: $< (-1)^n > = -1 , 1 , -1 , 1 , \dots$

$|x_n| = |(-1)^n| = 1 \implies < x_n >$ is bounded and $M = 1$

$< (-1)^n >$ is divergent?

Remake: If $< x_n >$ unbounded, then $< x_n >$ is divergent.

Proof:

Suppose that $< x_n >$ converged and unbounded sequence.

Since $< x_n >$ Convergent $\rightarrow < x_n >$ bounded by theorem (In metric space, every conv. Seq. is bounded) $\rightarrow$ C! ,So $< x_n >$ unbounded is $< x_n >$ is divergent

Ex:

- $< x_n > = < \sqrt{n-1} > = 0 , \sqrt{1} , \sqrt{2} , \sqrt{3} , \ldots$ unbounded $\Longrightarrow < x_n >$ divergent

- $< x_n > = < n^2 - n > = 0 , 2 , 6 , 11 , \ldots$ unbounded $\Longrightarrow < x_n >$ divergent

LCH (16)

Definition: Let $< x_n >$ be a real sequence. Then it is called

- Non – decreasing. If $x_{n+1} \geq x_n , \forall n$
- Non – increasing. If $x_{n+1} \leq x_n , \forall n$.
- Not monotone. If it does not increasing and decreasing.

Ex:

$* < x_n > = < \frac{1}{\sqrt{n}} >$

$x_n = \frac{1}{\sqrt{n}}$, $x_{n+1} = \frac{1}{\sqrt{n+1}}$

$\forall n, n+1 > n \implies \sqrt{n+1} > \sqrt{n} \rightarrow \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{2}} \rightarrow$

$x_{n+1} \leq x_n$

$\therefore < x_n >$ is non – increasing

$* < x_n > = < \frac{n}{n+1} >$

$x_n = \frac{n}{n+1}$, $x_{n+1} = \frac{n+1}{n+2}$

$x_{n+1} - x_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1) - n(n+2)}{(n+1)(n+2)} =$

$\frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$

$\therefore x_{n+1} - x_n > 0 \rightarrow x_{n+1} > x_n, \forall n, \therefore < x_n >$ non – decreasing

$* < x_n > = < (-1)^n >$ not monotone

$* < x_n > = < \frac{(-1)^n}{\sin(n)} >$ not monotone.

$* < x_n > = < (-5)^n >$ not monotone.

Theorem: Every monotone bounded real seq. is convergent

Ex: $< x_n > = \langle \frac{(-1)^n}{n} \rangle > 0$

$< x_n >$ Convergent seq. but not monotone.

Ex: Show that $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ is convergent.

Theorem: Let (X, d) be a metric space and $S \subseteq X$:

i. If $< x_n >$ seq. in S and $x_n \to x$ then $x \in S$ or $x \in S'$

ii. If $x \in S\ or\ x \in S'$, then there exists a sequence $< x_n >$ in S s.t $x_n \to x$

Definition: The sequence $< x_n >$ is a sub sequence of $< x_n >$, if $< m >$ is increasing sequence in N.

Ex: find a sub Seq. of the following seq.

1.$< x_n > = < \sqrt{n} >$

Solution:

$< \sqrt{n} > = \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots$

LEC (17)

Let $< m > = < 2n >$ increasing Seq. in N, the Sequence is

$$< Xm > = < \sqrt{2n} > = \sqrt{2}, \sqrt{4}, \sqrt{6}, \ldots$$

Let $< m > = < n + 3 >$ increasing seq in N, the sub seq is

$$< m > = < \sqrt{n+3} > = \sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots$$

Theorem: Let $< x_n >$ be a convergent Seq and $\lim_{n \to \infty} X_n = x$ then the sub seq $< X_{nm} >$ also conv. To x, where $n \to \infty$

Proof:

Since $x_n \to x, \forall \epsilon > 0, \exists k \in N\ s.t\ d(x_n, x) < \epsilon, \forall n > k$

Choose $nr > k$, then $\forall m > r \to nm > nr > k$

$\Longrightarrow d(x_{nm}, x) < \epsilon, \forall nm > k$

$\Longrightarrow < x_{nm} > \to x.$

Definition: Let (X, d) be a metrices space and $< x_n >$ be a seq. in X we say that $< x_n >$ is a principle. (Caushy) seq. if $\forall \epsilon > 0, \exists k \in N\ s.t\ d(x_n, x_m) < \epsilon, \forall n, m > k.$

Ex: prove that $< \frac{1}{n} >$ is Caushy seq in R?

Solution: $\forall\ \epsilon > 0$, to find $k \in N$ s.t $d(x_n, x_m) < \epsilon, \forall n, m > k, \forall n, m > k.$

Let $m > n \to d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

Since $\epsilon > 0$ (by Arch. Prop) $\to$ $\exists k \in N$ s.t

$k\epsilon > 2 \to \frac{2}{k} < \epsilon$

$\forall\, n > k\,, d(x_n, x_m) = |x_n - x_m| < \frac{2}{n} < \epsilon, \forall n, m > k \to < X_n >$ is Caushy seq.

Theorem: I metric space (X, d), every Convergent seq. is Caushy.

Remark: The Converse of the above theorem. Is not true by the following example.

Ex: Let $X = IR^{++}$ positive numbers $d(x, y) = |x - y|, \forall x, y \in R^{++}, \forall n > k$.

$< x_n > = < \frac{1}{n} >$ is Caushy seq.

But $\frac{1}{n} \to 0 \notin R^{++}$

$\therefore < \frac{1}{n} >$ is not Conv

Theorem: In metric Space (x, d) every Caushy seq. is bounded.

Ex: Let $< x_n > = (-1)^n$ be a seq.

$< x_n >$ is bounded seq, but not Caushy Seq

Since $d(-1,1) = 1 < \epsilon, \forall \epsilon > 0$

If $\epsilon = \frac{1}{2} \to 2 < \frac{1}{2} \to C!$

Theorem: For any real number $r, \exists$ rational Caushy Seq $< x_n >$ Conv to r.

LEC (18)

Definition: Let(X, d) be a metric space we say that X is Compete. If every Cauchy Seq.

In X coverage to a point in X.

i.e.: X is complete. If $\forall < X_n >$ Cauchy Seq.

$\to \exists \bar{x} \in X \; s.t \; X_n \to X$.

Theorem: Cantor's theorem for Nested sets.

Proof:

Let (X, d) be a Complete matric Space and $< E_n >$ be a seq of closed bounded Subset of X such that $E_1 \supset E_2 \supset \cdots E_n \supset E_{n+1} \; \forall n$ and the Sequence of Positive numbers $< \; daim \; E_n > \to 0$, then $\cap \; E_n =$ Singleton point

Remark: The condition of closed sets of Cantor’s theorem is necessary.

Ex: Let $E_n = \left(0\,,\frac{1}{n}\right)$ be the open intervals, $E_{n+1} \subset E_n$, and $daim(E_n) = \frac{1}{n} \to 0, \forall n$ E_n is bounded and not closed. Prove that $\cap\, E_n = \emptyset$

Proof:

Suppose $\cap\, E_n \neq \emptyset \to \ \exists r \in E_n\ s.t$

$r \in \left(0\,,\frac{1}{n}\right), \forall n$

Since $r > 0$, by Arch.pvop ,$\exists k \in N\ s.t$

$kr > 1 \to \frac{1}{k} < r \to C!$

➔ $\cap\, E_n = \emptyset$

Corollary: Let $< \pm n >$ be aseq of closed intervals, $I_n = [a_n, b_n]$ such that

1.$I_n \supset I_{n+1}$

2.$\lim_{n\to\infty} |I_n| = 0,$ then $\cap\, I_n$ =singleton Point

Theorem: R^n is Complete metric Space, $n \geq 1$

i.e.: (Every Cauchy sequence in R^n is Convergent)

Theorem: Let $< X_n >$, $< Y_n >$ and $< Z_n >$ real Sequence s.t $\forall n$, $X_n \leq Y_n \leq Z_n$ and

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} Z_n = a \text{ then}$$

$\lim_{n\to\infty} Y_n = a$

Theorem: let $< X_n >$ be a real sequence such that $< X_n >$ Converge to 0 and

$$X_n \geq 0 \, , p > 0 \text{ then } < X_n^p > \text{ converges to } 0$$

Proof:

$< X_n^p > = x_1^p , x_2^P , x_3^p , \ldots$

Since $< X_n > \to 0 \to \forall \epsilon > 0 \, , \exists \, k \in N \, s.t$

$|X_{n-0}| = |X_n| < \epsilon^p \, , \forall n > k$ and

$|X_n \, . X_n \, \ldots . X_n| = |X_n||X_n| \ldots . . |X_n| = |X_n|^p < \left(\epsilon^{\frac{1}{p}}\right)^p \, , \forall n > k$

$< X_n^p > \to 0$.

وفي الختام نسأل الله التوفيق

اللهم قني عذابك
يوم تبعث عبادك